

A Note on Central Limit Theorems for Linear Spectral Statistics of Large Dimensional F -matrix

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Abstract. Sample covariance matrix and multivariate F -matrix play important roles in multivariate statistical analysis. The central limit theorems (CLT) of linear spectral statistics associated with these matrices were established in Bai and Silverstein (2004) and Zheng (2012) which received considerable attentions and have been applied to solve many large dimensional statistical problems. However, the sample covariance matrices used in these papers are not centralized and there exist some questions about CLT's defined by the centralized sample covariance matrices. In this note, we shall provide some short complements on the CLT's in Bai and Silverstein (2004) and Zheng (2012), and show that the results in these two papers remain valid for the centralized sample covariance matrices, provided that the ratios of dimension p to sample sizes (n, n_1, n_2) are redefined as $p/(n-1)$ and $p/(n_i-1)$, $i=1, 2$, respectively.

Key words and phrases. Linear spectral statistics, central limit theorem, centralized sample covariance matrix, centralized F -matrix, simplified sample covariance matrix, simplified F -matrix.

1 Introduction

Let $\{X_{jk}, j, k = 1, 2, \dots\}$ and $\{Y_{jk}, j, k = 1, 2, \dots\}$ be two independent double arrays of independent random variables, either both real or both complex. In the sequel, we use A^* to denote a complex conjugate transpose of a vector or matrix \mathbf{A} . For $p > 1$, $n > 1$ and $N > 1$, we define $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_n)$ and $\mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_N)$ with column vectors $\mathbf{X}_j = (X_{j1}, \dots, X_{jp})'$, $1 \leq j \leq n$, and $\mathbf{Y}_k = (Y_{k1}, \dots, Y_{kp})'$, $1 \leq k \leq N$. Let \mathbf{T}_p be a $p \times p$ non-negative definite (nnd) matrix. There exists a unique nnd matrix $\mathbf{T}_p^{1/2}$ such that $\mathbf{T}_p = (\mathbf{T}_p^{1/2})^2$. Then, $(\mathbf{T}_p^{1/2}\mathbf{X}_1, \dots, \mathbf{T}_p^{1/2}\mathbf{X}_n)$ and $(\mathbf{T}_p^{1/2}\mathbf{Y}_1, \dots, \mathbf{T}_p^{1/2}\mathbf{Y}_N)$ can be considered as two independent samples of sizes n and N , respectively, drawn from a p -dimensional population with population covariance matrix \mathbf{T}_p .

It is well known that the sample covariance matrices for $\mathbf{T}_p^{1/2}\mathbf{X}$ and $\mathbf{T}_p^{1/2}\mathbf{Y}$ are often defined as

$$\mathbf{S}_x = \frac{1}{n-1} \left(\sum_{i=1}^n \mathbf{T}_p^{1/2} \mathbf{X}_i \mathbf{X}_i^* \mathbf{T}_p^{1/2} - n \mathbf{T}_p^{1/2} \bar{\mathbf{X}} \bar{\mathbf{X}}^* \mathbf{T}_p^{1/2} \right), \quad (1.1)$$

$$\mathbf{S}_y = \frac{1}{N-1} \left(\sum_{i=1}^N \mathbf{T}_p^{1/2} \mathbf{Y}_i \mathbf{Y}_i^* \mathbf{T}_p^{1/2} - N \mathbf{T}_p^{1/2} \bar{\mathbf{Y}} \bar{\mathbf{Y}}^* \mathbf{T}_p^{1/2} \right), \quad (1.2)$$

respectively, where $\bar{\mathbf{X}} = \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i$ and $\bar{\mathbf{Y}} = \frac{1}{N} \sum_{i=1}^N \mathbf{Y}_i$. The multivariate F -matrix¹ is then defined as

$$\mathbf{F} = \mathbf{S}_x \mathbf{S}_y^{-1}. \quad (1.3)$$

Notice that the matrices defined in (1.1)–(1.3) are transformation invariant, we will call them *centralized* sample covariance matrices and multivariate F -matrix, respectively.

Due to Corollary A.41 and Theorem A.43 of Bai and Silverstein (2009), in the literature of random matrix theory, the sample covariance matrices are usually *simplified* as

$$\mathbf{B}_x = \frac{1}{n} \sum_{i=1}^n \mathbf{T}_p^{1/2} \mathbf{X}_i \mathbf{X}_i^* \mathbf{T}_p^{1/2} \text{ and } \mathbf{B}_y = \frac{1}{N} \sum_{i=1}^N \mathbf{T}_p^{1/2} \mathbf{Y}_i \mathbf{Y}_i^* \mathbf{T}_p^{1/2} \quad (1.4)$$

and the multivariate F -matrix is *simplified* as $\mathbf{G} = \mathbf{B}_x \mathbf{B}_y^{-1}$.

Bai and Silverstein (2004) considered the central limit theorem (CLT) of the linear spectral statistics (LSS) of the simplified sample covariance matrix \mathbf{B}_x and provided the explicit expressions of asymptotic means and covariance functions for \mathbf{B}_x . Later, Zheng (2012) extended the work of Bai and Silverstein (2004) to the case of the multivariate F -matrix \mathbf{G} and obtained explicit expressions of the asymptotic means, variances, and covariances for \mathbf{G} .

Examining the inequalities derived from Corollary A.41 and Theorem A.43 of Bai and Silverstein (2009), one finds that the difference between the empirical spectral distributions (*ESD*) of \mathbf{S}_x and \mathbf{B}_x is of the order $O(n^{-1})$. Hence, we conclude that \mathbf{S}_x and \mathbf{B}_x have the same limiting spectral distributions (*LSD*). However, the scale normalizers in CLT's of LSS of random matrices \mathbf{S}_x and \mathbf{B}_x have the same order as p . Thus, it is expected that the asymptotic biases in the CLT's of LSS of \mathbf{S}_x and \mathbf{B}_x should have a little difference. Upon such a consideration, Pan (2012) reconsidered the CLT of LSS of centralized sample covariance matrix \mathbf{S}_x . To reduce the asymptotic bias, he added an additional term to that of Bai and Silverstein (2004), that is,

$$\frac{y}{2\pi i} \int g(z) \frac{\underline{m}_y(z) \int \frac{tdH(t)}{(1+t\underline{m}_y(z))^2}}{z \left(1 - y \int \frac{\underline{m}_y^2(z)t^2 dH(t)}{(1+t\underline{m}_y(z))^2} \right)} dz, \quad (1.5)$$

¹To guarantee that the definition makes sense, we need to assume that $p < N$ and \mathbf{T}_p is positive definite. Because the eigenvalues of \mathbf{F} are independent of \mathbf{T}_p , we may assume \mathbf{T}_p is an identity matrix.

where $\underline{m}_y(z) = -\frac{1-y}{z} + ym_y(z)$, $m_y(z)$ is the Stieltjes transform of the LSD of \mathbf{S}_x , $H(t)$ is the LSD of \mathbf{T}_p and $y_n = p/n \rightarrow y > 0$.

It is well known that when the population is multivariate-normally distributed, the centralized sample covariance matrix \mathbf{S}_x has the same distribution as simplified covariance matrix \mathbf{B}_x with sample size $n - 1$ and population mean zero. This fact motivates that this phenomenon should be asymptotically true in the general case. In this note, we shall give short proofs to indicate that if the simplified sample covariance matrix \mathbf{B}_x is replaced by centralized sample covariance matrix \mathbf{S}_x , Bai and Silverstein (2004)'s result remains valid provided that the ratio of dimension to sample size y_n is replaced by $p/(n - 1)$ (this is equivalent to $c_n = n/(N - 1)$ in Bai and Silverstein (2004)). This result is equivalent to but much simpler than that of Pan (2012) in both expressions and proof. Moreover, we shall prove that if the simplified multivariate F -matrix \mathbf{G} is replaced by the centralized \mathbf{F} , the results of Zheng (2012) remain valid provided the ratios of dimensions to sample sizes, y_{n1} and y_{n2} , are replaced by $p/(n - 1)$ and $p/(N - 1)$.

The remainder of this note is arranged as follows: Section 2 states the main theorems and the proof of Theorem 2.2. Section 3 gives the proof of Theorem 2.1. The technical lemmas and their proofs will be postponed to Section 4.

2 Main Results

As mentioned in the previous section, the centralized covariance matrix will have the same LSD as that of the corresponding simplified covariance matrix. In this note, we shall prove the following theorems.

Theorem 2.1 *Assume that*

(a) *For each p , $\{X_{ij}, i \leq p, j \leq n\}$ are independent random variables with $EX_{ij} = 0$, $E|X_{11}|^2 = 1$, and satisfying*

$$\frac{1}{np} \sum_{j=1}^p \sum_{k=1}^n E|X_{jk}|^4 \mathbf{1}_{\{|X_{jk}| \geq \eta\sqrt{n}\}} \rightarrow 0, \quad \text{for any fixed } \eta > 0. \quad (2.1)$$

Note that the random variables may be allowed to depend on p , but we suppress this dependence from the notation for brevity.

(b) *We assume $E|X_{ij}|^4 = 3$ for the real case, and $E|X_{ij}|^4 = 2$ and $EX_{ij}^2 = 0$ for the complex case.*

(c) *$y_n = p/n \rightarrow y$, and*

(d) *\mathbf{T}_p is a $p \times p$ non-random nnd Hermitian matrix with bounded spectral norm in p , and its ESD $H_p \xrightarrow{D} H$*

where H is a proper probability distribution.

Let f be an analytic function on an open region in the complex plane which covers the support of LSD of \mathbf{S}_x with the origin excluded.

Then

(i) the random variables

$$X_p(f) = p \int f(x) d\left(F^{\mathbf{S}_x} - F^{\{y_{n-1}, H_p\}}(x)\right), \quad (2.2)$$

form a tight sequence in p , where $F^{\mathbf{S}_x}$ is the ESD of centralized sample covariance matrix \mathbf{S}_x , $F^{\{y, H\}}$ is the LSD of \mathbf{S}_x whose LSD's Stieltjes transform $m_y(z)$ satisfies $\underline{m}_y(z) = ym_y(z) - (1-y)/z$ and $\underline{m}_y(z)$ is the unique solution to the equation

$$z = -\frac{1}{\underline{m}_y} + y \int \frac{t}{1 + t\underline{m}_y(z)} dH(t). \quad (2.3)$$

in the upper half complex plane for each $z \in \mathbb{C}^+ = \{z : \Im(z) > 0\}$.

(ii) The random variables in (2.2) converges weakly to Gaussian variables X_f with the same means and covariance functions as given in Theorem 1.1 of Bai and Silverstein (2004).

The proof of Theorem 2.1 is postponed to Section 3.

As for the CLT of LSS of \mathbf{F} matrix, we have the following theorem.

Theorem 2.2 Assume that

1. the two arrays $\{X_{jk}, j \leq p, k \leq n\}$ and $\{Y_{jk}, j \leq p, k \leq N\}$ satisfy for any fixed $\eta > 0$,

$$\frac{1}{np} \sum_{j=1}^p \sum_{k=1}^n E|X_{jk}|^4 \mathbf{1}_{\{|X_{jk}| \geq \eta\sqrt{n}\}} \rightarrow 0, \quad \frac{1}{Np} \sum_{j=1}^p \sum_{k=1}^N E|Y_{jk}|^4 \mathbf{1}_{\{|Y_{jk}| \geq \eta\sqrt{N}\}} \rightarrow 0. \quad (2.4)$$

2. For all j, k , $|EX_{jk}^4| = \beta_x + 1 + \kappa$, $|EY_{jk}^4| = \beta_y + 1 + \kappa$. If both \mathbf{X} and \mathbf{Y} are complex valued, then

$$EX_{jk}^2 = EY_{jk}^2 = 0. \text{ Moreover, } y_n = p/n \rightarrow y_1 > 0 \text{ and } y_N = p/N \rightarrow y_2 \in (0, 1).$$

Let f be an analytic function in an open region of the complex plane containing the interval $\left[\frac{(1-h)^2}{(1-y_2)^2}, \frac{(1+h)^2}{(1-y_2)^2}\right]$, the support of the continuous part of the LSD $F_{\mathbf{Y}}$ of \mathbf{F} -matrix, $h = \sqrt{y_1 + y_2 - y_1 y_2}$ and $\mathbf{y} = (y_1, y_2)$.

Then, as $p \rightarrow \infty$, the random variables

$$W_p(f) = p \int f(x) d\left(F^{\mathbf{F}}(x) - F_{(y_{n-1}, y_{N-1})}(x)\right)$$

converges weakly to Gaussian variables $\{W_f\}$ which have the same means and covariance functions as given in Zheng (2012), where $F^{\mathbf{F}}(x)$ is the ESD of centralized F -matrix \mathbf{F} and $F_{(y_1, y_2)}(x)$ is the LSD defined by (2.4) of Zheng (2012).

Proof. As mentioned in Section 1, we may assume \mathbf{T}_p to be an identity matrix. Split our proofs into two steps by writing

$$tr(\mathbf{F} - z\mathbf{I}_p)^{-1} - pm_{(y_{n-1}, y_{N-1})}(z) = \left[tr(\mathbf{S}_x \mathbf{S}_y^{-1} - z\mathbf{I}_p)^{-1} - pm^{(y_{n-1}, F^{\mathbf{S}_y^{-1}})}(z) \right] + p \left[m^{(y_{n-1}, F^{\mathbf{S}_y^{-1}})}(z) - m_{(y_{n-1}, y_{N-1})}(z) \right]$$

where $F^{\mathbf{S}_y^{-1}}(t)$ and $F^{\mathbf{S}_y}(t)$ are the ESDs of \mathbf{S}_y^{-1} and \mathbf{S}_y , $m_{(y_1, y_2)}$ is the Stieltjes transform of the LSD of \mathbf{F} matrix,

$$\begin{aligned} \underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}} &= -\frac{1 - y_{n-1}}{z} + y_{n-1} m^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}}(z) \\ z &= -\frac{1}{\underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}}} + y_{n-1} \int \frac{t}{1 + t \underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}}} dF^{\mathbf{S}_y^{-1}}(t) \\ &= -\frac{1}{\underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}}} + y_{n-1} \int \frac{1}{t + \underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}}} dF^{\mathbf{S}_y}(t). \end{aligned} \quad (2.5)$$

Step 1. Given \mathbf{S}_y , in the proof of Theorem 2.1, we have proved that the process $tr(\mathbf{S}_x \mathbf{S}_y^{-1} - z\mathbf{I}_p)^{-1} - pm^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}}(z)$ weakly tends to a Gaussian process on the contour with mean and covariance function as given in (6.29) and (6.30) of Zheng (2012).

Step 2. By (2.5) and the truth of

$$z = -\frac{1}{\underline{m}_{\{y_{n-1}, y_{N-1}\}}} + y_{n-1} \int \frac{1}{t + \underline{m}_{\{y_{n-1}, y_{N-1}\}}} dF_{y_{N-1}}(t), \quad (2.6)$$

where $F_{y_{N-1}}$ is the M-P law with ratio of dimension to sample size y_{N-1} . Subtracting both sides of (2.5) from those of (2.6), we obtain

$$\begin{aligned} p \cdot \left[m^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}}(z) - m_{\{y_{n-1}, y_{N-1}\}}(z) \right] &= N \cdot \left[\underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}}(z) - \underline{m}_{\{y_{n-1}, y_{N-1}\}}(z) \right] \\ &= -y_{n-1} \underline{m}_{\{y_{n-1}, y_{N-1}\}} \underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}} \frac{tr \left(\mathbf{S}_y + \underline{m}_{\{y_{n-1}, y_{N-1}\}} \mathbf{I}_p \right)^{-1} - pm_{y_{N-1}}(-\underline{m}_{\{y_{n-1}, y_{N-1}\}})}{1 - y_{n-1} \cdot \int \frac{\underline{m}_{\{y_{n-1}, y_{N-1}\}} \cdot \underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}} dF_{N-1}(t)}{(t + \underline{m}_{\{y_{n-1}, y_{N-1}\}}) \cdot (t + \underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}})} \\ &= -y_{n-1} \underline{m}_{\{y_{n-1}, y_{N-1}\}} \underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}} \frac{p[m_{N-1}(-\underline{m}_{\{y_{n-1}, y_{N-1}\}}) - m_{y_{N-1}}(-\underline{m}_{\{y_{n-1}, y_{N-1}\}})]}{1 - y_{n-1} \cdot \int \frac{\underline{m}_{\{y_{n-1}, y_{N-1}\}} \cdot \underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}} dF_{N-1}(t)}{(t + \underline{m}_{\{y_{n-1}, y_{N-1}\}}) \cdot (t + \underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}})} \\ &= \frac{-y_{n-1} \underline{m}_{\{y_{n-1}, y_{N-1}\}} \underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}} \cdot p[m_{N-1}(-\underline{m}_{\{y_{n-1}, y_{N-1}\}}) - m_{y_{N-1}}(-\underline{m}_{\{y_{n-1}, y_{N-1}\}})]}{1 - y_{n-1} \cdot \int \frac{\underline{m}_{\{y_{n-1}, y_{N-1}\}} \cdot \underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}} dF_{N-1}(t)}{(t + \underline{m}_{\{y_{n-1}, y_{N-1}\}}) \cdot (t + \underline{m}^{\{y_{n-1}, F^{\mathbf{S}_y^{-1}}\}})} \end{aligned} \quad (2.7)$$

which weakly tends to a Gaussian process on the contour with mean and covariance function as given in (6.33) and (6.34) of Zheng (2012) where m_{N-1} is the Stieltjes transform of ESD $F_{N-1}(x)$ of \mathbf{S}_y , $z = -\frac{1}{\underline{m}_{y_{N-1}}} + \frac{y_{N-1}}{1+\underline{m}_{y_{N-1}}}$ and $\underline{m}_{y_{N-1}}(z) = -\frac{1-y_{N-1}}{z} + y m_{y_{N-1}}(z)$. By Theorem 2.1 and (2.7) we obtain that

$$\text{tr}(\mathbf{F} - z\mathbf{I}_p)^{-1} - pm_{\{y_{n-1}, y_{N-1}\}}(z) \quad \text{and} \quad \text{tr}(\mathbf{G} - z\mathbf{I}_p)^{-1} - pm_{\{y_n, y_N\}}(z)$$

have the same asymptotic distribution. Hence, the random variables

$$\left(p \int f(x) d(F^{\mathbf{F}}(x) - F_{\{y_{n-1}, y_{N-1}\}}(x)) \right)$$

converges weakly to Gaussian variables W_f with the same means and covariance functions as Zheng (2012).

Then the proof of Theorem 2.2 is completed. ■

3 The Proof of Theorem 2.1

The condition (2.1) allows us to truncate the random variables at $\eta_n \sqrt{n}$ and then renormalize them to have means zero and variances 1, where $\eta_n \rightarrow 0$ slowly. Note that the 4th moments of the random variables may not be the same but they will be $\kappa + 1 + \beta_x + o(1)$ and for the complex case we have $EX_{ij}^2 = o(n^{-1})$. The contour is defined similar to Bai and Silverstein (2004). Define $\gamma_i = \frac{1}{\sqrt{n}} \mathbf{T}_p^{1/2} \mathbf{X}_i$. Then, we have

$$\mathbf{S}_x = \frac{n}{n-1} \sum_{i=1}^n (\gamma_i - \bar{\gamma})(\gamma_i - \bar{\gamma})^* = \sum_{i=1}^n \gamma_i \gamma_i^* - \frac{1}{n-1} \sum_{i \neq j} \gamma_i \gamma_j^* = \mathbf{B}_x - \Delta$$

where $\bar{\gamma} = \frac{1}{n} \sum_{i=1}^n \gamma_i$ and $\Delta = \frac{1}{n-1} \sum_{j \neq k} \gamma_j \gamma_k^*$. Because

$$(\mathbf{S}_x - z\mathbf{I})^{-1} = (\mathbf{A} - \Delta)^{-1} = \mathbf{A}^{-1} + (\mathbf{A} - \Delta)^{-1} \Delta (\mathbf{A} - \Delta)^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1} \Delta \mathbf{A}^{-1} + \mathbf{A}^{-1} (\Delta \mathbf{A}^{-1})^2 + (\mathbf{A} - \Delta)^{-1} (\Delta \mathbf{A}^{-1})^3$$

where $\mathbf{A}(z) = \mathbf{B}_x - z\mathbf{I}$, then we obtain

$$\begin{aligned} & p \left(\frac{1}{p} \text{tr}(\mathbf{B}_x - z\mathbf{I})^{-1} - m_{n-1}^0(z) \right) \\ &= p \left(\frac{1}{p} \text{tr}(\mathbf{A} - \Delta)^{-1} - m_n^0(z) + m_n^0(z) - m_{n-1}^0(z) \right) \\ &= p \left(\frac{1}{p} \text{tr} \mathbf{A}^{-1}(z) - m_n^0(z) \right) + p(m_n^0(z) - m_{n-1}^0(z)) + \text{tr} \mathbf{A}^{-2}(z) \Delta \\ & \quad + \text{tr} \mathbf{A}^{-1}(z) (\Delta \mathbf{A}^{-1}(z))^2 + \text{tr} (\mathbf{A}(z) - \Delta)^{-1} (\Delta \mathbf{A}^{-1}(z))^3 \end{aligned} \tag{3.1}$$

where $\underline{m}_n^0(z) = -\frac{1-y_n}{z} + y_n m_n^0(z)$, $\underline{m}_n^0(z)$ and $\underline{m}_{n-1}^0(z)$ satisfy

$$z = -\frac{1}{\underline{m}_n^{(0)}} + \frac{p}{n} \int \frac{t}{1 + t \underline{m}_n^{(0)}} dH_p(t) \tag{3.2}$$

$$z = -\frac{1}{\underline{m}_{n-1}^{(0)}} + \frac{p}{n-1} \int \frac{t}{1+t\underline{m}_{n-1}^{(0)}} dH_p(t) \quad (3.3)$$

$$z = -\frac{1}{\underline{m}_y} + y \int \frac{t}{1+t\underline{m}_y} dH(t). \quad (3.4)$$

By (3.1), Lemmas 4.1 and 4.6, we have

$$\text{tr}(\mathbf{B}_x - z\mathbf{I})^{-1} - p \cdot m_{n-1}^0(z) = p \left(\frac{1}{p} \text{tr} \mathbf{A}^{-1}(z) - m_n^0(z) \right) + o_p(1). \quad (3.5)$$

So Theorem 2.1 has the same asymptotic mean and covariance function as Theorem 1.1 of Bai and Silverstein (2004).

Tightness of $\text{tr}(\mathbf{S}_x - z\mathbf{I})^{-1} - \text{tr}(\mathbf{B}_x - z\mathbf{I})^{-1}$. Using what has been proved in Bai and Silverstein (2004), we only need to prove that there is an absolute constant M such that for any $z_1, z_2 \in \mathcal{C}$,

$$\begin{aligned} & \mathbb{E} \left| \text{tr}(\mathbf{B}_x - z_1\mathbf{I})^{-1}(\mathbf{B}_x - z_2\mathbf{I})^{-1} - \text{tr}(\mathbf{S}_x - z_1\mathbf{I})^{-1}(\mathbf{S}_x - z_2\mathbf{I})^{-1} \right|^2 \\ &= \mathbb{E} \left| \sum_{i=1}^n \frac{(\lambda_i - \tilde{\lambda}_i)(\lambda_i + \tilde{\lambda}_i - 2z_1z_2)}{(\lambda_i - z_1)(\lambda_i - z_2)(\tilde{\lambda}_i - z_1)(\tilde{\lambda}_i - z_2)} \right|^2 \\ &\leq K \mathbb{E} \left| \sum_{i=1}^n |\lambda_i - \tilde{\lambda}_i| \right|^2 \mathcal{B}_n + o(1) \leq M, \end{aligned} \quad (3.6)$$

where $\{\lambda_i\}$ and $\{\tilde{\lambda}_i\}$ are the eigenvalues of \mathbf{B}_x and \mathbf{S}_x , respectively, and arranged in descending order, the event \mathcal{B}_n is defined as $x_l + \epsilon < \tilde{\lambda}_p < \lambda_1 < x_r - \epsilon$ such that

$$\mathbb{P}(\mathcal{B}_n) = o(n^{-3}).$$

(for the justification of the definition \mathcal{B}_n , see Bai and Silverstein (1998)). The last step of (3.6) follows from the fact that

$$\sum_{i=1}^n |\lambda_i - \tilde{\lambda}_i| = \sum_{i=1}^n (\lambda_i - \tilde{\lambda}_i) \leq \lambda_1 - \tilde{\lambda}_p \leq x_r$$

by the interlacing theorem.

The equi-continuity of $\text{Etr}(\mathbf{S}_x - z\mathbf{I})^{-1} - pm_{n-1}^0(z)$ can be proved in a similar way to that for the tightness of $\text{tr}(\mathbf{S}_x - z\mathbf{I})^{-1} - \text{Etr}(\mathbf{B}_x - z\mathbf{I})^{-1}$.

By now, the proof of Theorem 2.1 is completed. ■

4 Technical Lemmas

Lemma 4.1 *Under assumptions of Theorem 2.1, for every $z \in \mathbb{C}^+$, we have*

$$p(m_n^{(0)} - m_{n-1}^{(0)}) \rightarrow (1 + z\underline{m}_y) \cdot \frac{\underline{m}_y + z\underline{m}'_y}{z\underline{m}_y}.$$

Proof. We have

$$\underline{m}_n^{(0)}(z) = -\frac{n-p}{n} \cdot \frac{1}{z} + \frac{p}{n} m_n^0(z), \quad \underline{m}_{n-1}^{(0)}(z) = -\frac{n-1-p}{n-1} \cdot \frac{1}{z} + \frac{p}{n} m_{n-1}^0(z) \quad (4.1)$$

where $p/n \rightarrow y > 0$. By (3.4), we obtain

$$\underline{m}'_y = \frac{1}{\frac{1}{\underline{m}_y^2} - y \int \frac{t^2}{(1+t\underline{m}_y)^2} dH(t)}, \quad y \int \frac{t}{1+t\underline{m}_y} dH(t) = \frac{1+z\underline{m}_y}{\underline{m}_y} \quad (4.2)$$

Using (3.2)-(3.3), we obtain

$$0 = \frac{\underline{m}_n^{(0)} - \underline{m}_{n-1}^{(0)}}{\underline{m}_n^{(0)} \underline{m}_{n-1}^{(0)}} - (\underline{m}_n^{(0)} - \underline{m}_{n-1}^{(0)}) \frac{p}{n} \int \frac{t^2}{(1+t\underline{m}_n^{(0)})(1+t\underline{m}_{n-1}^{(0)})} dH_p(t) - \frac{p}{n(n-1)} \int \frac{t}{1+t\underline{m}_{n-1}^{(0)}} dH_p(t),$$

that is,

$$n(\underline{m}_n^{(0)} - \underline{m}_{n-1}^{(0)}) = \frac{\frac{p}{n-1} \int \frac{t}{1+t\underline{m}_{n-1}^{(0)}} dH_p(t)}{\frac{1}{\underline{m}_n^{(0)} \underline{m}_{n-1}^{(0)}} - \frac{p}{n} \int \frac{t^2}{(1+t\underline{m}_n^{(0)})(1+t\underline{m}_{n-1}^{(0)})} dH_p(t)} \rightarrow \frac{y \int \frac{t}{1+t\underline{m}_y} dH(t)}{\frac{1}{\underline{m}_y^2} - y \int \frac{t^2}{(1+t\underline{m}_y)^2} dH(t)}. \quad (4.3)$$

By (4.1), (4.2) and (4.3), we have

$$\begin{aligned} p(m_n^{(0)} - m_{n-1}^{(0)}) &= n\underline{m}_n^{(0)} + \frac{n-p}{z} - \left((n-1)\underline{m}_{n-1}^{(0)} + \frac{n-1-p}{z} \right) \\ &= n(\underline{m}_n^{(0)} - \underline{m}_{n-1}^{(0)}) + \underline{m}_{n-1}^{(0)}(z) + \frac{1}{z} \\ &\rightarrow \frac{y \int \frac{t}{1+t\underline{m}_y} dH(t)}{\frac{1}{\underline{m}_y^2} - y \int \frac{t^2}{(1+t\underline{m}_y)^2} dH(t)} + \frac{1+z\underline{m}_y(z)}{z} \\ &= \underline{m}_y^* \cdot \frac{1+z\underline{m}_y}{\underline{m}_y} + \frac{1+z\underline{m}_y(z)}{z} = (1+z\underline{m}_y) \cdot \frac{\underline{m}_y + z\underline{m}'_y}{z\underline{m}_y}. \end{aligned} \quad (4.4)$$

Thus, we prove that Lemma 4.1 holds. ■

In the sequel, we shall use Vatali lemma frequently. Let

$$\Delta = \frac{1}{n} \sum_{j \neq k} \gamma_j \gamma_k^*. \text{ (It should be } \frac{1}{n-1} \sum_{j \neq k} \gamma_j \gamma_k^* \text{ but no harm to the limit.)}$$

We will derive the limit $\text{tr}(\mathbf{A} - \Delta)^{-1} - \text{tr}(\mathbf{A}^{-1})$.

Lemma 4.2 *After truncation and normalization, we have $E|\gamma_k^* \mathbf{A}^{-1} \gamma_k - (1+z\underline{m}_y(z))|^2 \leq Kn^{-1}$ for every $z \in \mathbb{C}^+$.*

Proof. We have $\gamma_k^* \mathbf{A}^{-1} \gamma_k = \gamma_k^* \mathbf{A}_k^{-1} \gamma_k \beta_k = 1 - \beta_k$, where $\mathbf{A}_k = \mathbf{A} - \gamma_k \gamma_k^*$ and $\beta_k + (1 + \gamma_k^* \mathbf{A}_k^{-1} \gamma_k)^{-1}$. Therefore,

By (1.15) and (2.17) of Bai and Silverstein (2004), we have $E|\gamma_k^* \mathbf{A}^{-1} \gamma_k - g(z)|^2 = E|\beta_k + z\underline{m}_y(z)|^2 \leq Kn^{-1}$ with

$g(z) = 1 + z\underline{m}_y(z)$.

Corollary 4.1 *After truncation and normalization, we have $\mathbb{E} \left| \gamma_k^* \mathbf{A}^{-2} \gamma_k - \frac{d}{dz}(1 + z \underline{m}_y(z)) \right|^2 \leq K n^{-1}$ for every $z \in \mathbb{C}^+$.*

Proof. By Cauchy integral formula, we have

$$\gamma_k^* \mathbf{A}^{-2} \gamma_k = \frac{1}{2\pi i} \oint_{|\zeta - z| = v/2} \frac{\gamma_k^* \mathbf{A}^{-1}(\zeta) \gamma_k}{(\zeta - z)^2} d\zeta \quad \text{and} \quad g'(z) = \frac{1}{2\pi i} \oint_{|\zeta - z| = v/2} \frac{1 + \zeta \underline{m}_y(\zeta)}{(\zeta - z)^2} d\zeta.$$

Then $\mathbb{E} \left| \gamma_k^* \mathbf{A}^{-2} \gamma_k - \frac{d}{dz}(1 + z \underline{m}_y(z)) \right|^2 \leq K n^{-1}$ follows from Lemma 4.2.

Lemma 4.3 *For any subset \mathcal{U} of $\{1, 2, \dots, n\}$, after truncating and normalizing, we have $\mathbb{E} |\text{tr} \mathbf{A}^{-1} \Delta|^2 \leq K n^{-1}$ for every $z \in \mathbb{C}^+$. Especially for every $z \in \mathbb{C}^+$,*

$$\mathbb{E} |\text{tr}(\mathbf{A}^{-2} \Delta)|^2 = \mathbb{E} \left| \frac{1}{n} \sum_{j \neq k} \gamma_j^* \mathbf{A}^{-2} \gamma_k \right|^2 = O(n^{-1}).$$

Proof. We have $\text{tr} \mathbf{A}^{-1} \Delta = \frac{1}{n} \sum_{j \neq k \in \mathcal{U}} \gamma_j^* \mathbf{A}^{-1} \gamma_k = \frac{1}{n} \sum_{j \neq k \in \mathcal{U}} \gamma_j^* \mathbf{A}_{jk}^{-1} \gamma_k \beta_j \beta_{k(j)}$, where $\mathbf{A}_{jk} = \mathbf{A}_k - \gamma_j \gamma_j^*$ for $j \neq k$ and $\beta_{k(j)} = (1 + \gamma_k^* \mathbf{A}_{jk}^{-1} \gamma_k)^{-1}$. We will similarly define \mathbf{A}_{ijk} and $\beta_{k(ij)}$ for later use. Then we obtain

$$\begin{aligned} \mathbb{E} |\text{tr}(\mathbf{A}^{-1} \Delta)|^2 &= \mathbb{E} \frac{1}{n} \sum_{j_1 \neq k_1 \in \mathcal{U}} \gamma_{j_1}^* \mathbf{A}_{j_1 k_1}^{-1} \gamma_{k_1} \beta_{j_1} \beta_{k_1(j_1)} \frac{1}{n} \sum_{j_2 \neq k_2 \in \mathcal{U}} \gamma_{k_2}^* \mathbf{A}_{j_2 k_2}^{-1} \gamma_{j_2} \beta_{j_2} \beta_{k_2(j_2)} \\ &:= \sum_{(2)} + \sum_{(3)} + \sum_{(4)}, \end{aligned}$$

where the index (\cdot) denotes the number of distinct integers in the set $\{j_1, k_1, j_2, k_2\}$. By the facts that $|\beta_j| \leq \frac{|z|}{\nu}$ and $\nu = \Im(z)$, we have

$$\begin{aligned} \sum_{(2)} &\leq \frac{2|z|^4}{n^2 v^4} \sum_{j \neq k \in \mathcal{U}} \mathbb{E} |\gamma_j^* \mathbf{A}_{jk}^{-1} \gamma_k|^2 \\ &\leq \frac{|z|^4}{\nu^4 n^4} \sum_{j \neq k} \mathbb{E} |\text{tr}(\mathbf{T}^* \mathbf{A}_{jk}^{-1} \mathbf{T} \bar{\mathbf{A}}_{jk}^{-1})| \leq \frac{p}{n^2} \frac{|z|^4 \|\mathbf{T}\|^2}{\nu^6} \leq K n^{-1} \end{aligned}$$

$$\sum_{(4)} = \frac{1}{n^2} \sum_{j_1 \neq k_1 \neq j_2 \neq k_2 \in \mathcal{U}} \mathbb{E} \gamma_{j_1}^* \mathbf{A}_{j_1 k_1}^{-1} \gamma_{k_1} \gamma_{j_2}^* \mathbf{A}_{j_2 k_2}^{-1} \gamma_{k_2} \beta_{j_1} \beta_{k_1(j_1)} \beta_{j_2} \beta_{k_2(j_2)}$$

where

$$\begin{aligned} \gamma_{j_1}^* \mathbf{A} \gamma_{k_1} &= \beta_{j_1} \beta_{k_1(j_1)} \gamma_{j_1}^* \mathbf{A}_{j_1 k_1}^{-1} \gamma_{k_1} = \beta_{j_1} \beta_{k_1(j_1)} \left[\gamma_{j_1}^* \mathbf{A}_{j_1 k_1 k_2}^{-1} \gamma_{k_1} - \beta_{k_2(j_1 k_1)} \gamma_{j_1}^* \mathbf{A}_{j_1 k_1 k_2}^{-1} \gamma_{k_2} \gamma_{k_2}^* \mathbf{A}_{j_1 k_1 k_2}^{-1} \gamma_{k_1} \right] \\ &= \beta_{j_1} \beta_{k_1(j_1)} \left[\gamma_{j_1}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{k_1} - \beta_{j_2(j_1 k_1 k_2)} \gamma_{j_1}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{j_2} \gamma_{j_2}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{k_1} \right. \\ &\quad - \beta_{k_2(j_1 k_1)} \gamma_{j_1}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{k_2} \gamma_{k_2}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{k_1} \\ &\quad + \beta_{k_2(j_1 k_1)} \beta_{j_2(j_1 k_1 k_2)} \gamma_{j_1}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{j_2} \gamma_{j_2}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{k_2} \gamma_{k_2}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{k_1} \\ &\quad + \beta_{k_2(j_1 k_1)} \beta_{j_2(j_1 k_1 k_2)} \gamma_{j_1}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{k_2} \gamma_{k_2}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{j_2} \gamma_{j_2}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{k_1} \\ &\quad \left. - \beta_{k_2(j_1 k_1)} \beta_{j_2(j_1 k_1 k_2)}^2 \gamma_{j_1}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{j_2} \gamma_{j_2}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{k_2} \gamma_{k_2}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{j_2} \gamma_{j_2}^* \mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} \gamma_{k_1} \right] \end{aligned}$$

and

$$\begin{aligned}\beta_j &= b_j - \beta_j b_j \epsilon_j = b_j - b_j^2 \epsilon_j + \beta_j b_j^2 \epsilon_j^2 \\ \beta_{j(k)} &= b_{j(k)} - \beta_{j(k)} b_{j(k)} \epsilon_{j(k)} = b_{j(k)} - b_{j(k)}^2 \epsilon_{j(k)} + \beta_{j(k)} b_{j(k)}^2 \epsilon_{j(k)}^2\end{aligned}\quad (4.5)$$

with $b_j = \frac{1}{1+E\gamma_j^* \mathbf{A}_j \gamma_j}$, $\epsilon_j = \gamma_j^* \mathbf{A}_j \gamma_j - E\gamma_j^* \mathbf{A}_j \gamma_j$, and $b_{j(k)}$ and $\epsilon_{j(k)}$ are similarly defined by replacing \mathbf{A}_j^{-1} as \mathbf{A}_{jk}^{-1} .

By the same manner, we can decompose $\gamma_{j_2}^* \mathbf{A} \gamma_{k_2}$ into similar 6 terms and then we will estimate the expectations of the 36 products in the expansion of $\gamma_{j_1}^* \mathbf{A} \gamma_{k_1} (\gamma_{j_2}^* \mathbf{A} \gamma_{k_2})^*$.

Case 1. There are at least 6 terms of $\mathbf{A}_{j_1 j_2 k_1 k_2}^{-1} := \mathbf{B}$'s contained in the product. We shall use the fact that all β -factors are bounded $|z|/v \leq K$. Then we can show that the term is bounded by $O(n^{-3})$. Say, for the product of the two 6-th terms, its expectation is bounded by

$$\begin{aligned}& KE \left| (\gamma_{j_1}^* \mathbf{B} \gamma_{j_2} \gamma_{j_2}^* \mathbf{B} \gamma_{k_2} \gamma_{k_2}^* \mathbf{B} \gamma_{j_2} \gamma_{j_2}^* \mathbf{B} \gamma_{k_1}) (\gamma_{j_2}^* \mathbf{B} \gamma_{j_1} \gamma_{j_1}^* \mathbf{B} \gamma_{k_1} \gamma_{k_1}^* \mathbf{B} \gamma_{j_1} \gamma_{j_1}^* \mathbf{B} \gamma_{k_2})^* \right| \\ & \leq K \left(\mathbb{E} \left| (\gamma_{j_1}^* \mathbf{B} \gamma_{j_2} \gamma_{j_2}^* \mathbf{B} \gamma_{k_1} \gamma_{k_2} \gamma_{j_2}^* \mathbf{B} \gamma_{j_2} \gamma_{j_2}^* \mathbf{B} \gamma_{k_1}) \right|^2 \mathbb{E} \left| (\gamma_{j_2}^* \mathbf{B} \gamma_{j_1} \gamma_{j_2}^* \mathbf{B} \gamma_{k_2} \gamma_{k_1}^* \mathbf{B} \gamma_{j_1} \gamma_{j_2}^* \mathbf{B} \gamma_{k_2}) \right|^2 \right)^{1/2}.\end{aligned}$$

Note that the factors $\gamma_{j_2}^* \mathbf{B} \gamma_{k_2}$ in the first batch and $(\gamma_{j_1}^* \mathbf{B} \gamma_{k_1})^*$ in the second batch are exchanged positions when using the Cauchy-Schwarz for avoiding 8th power of the γ under the expectation sign.

Applying the formula

$$\begin{aligned}& \mathbb{E} \left| \sum_{i=1}^n \bar{X}_i Y_i \sum_{j=1}^n \bar{X}_j Z_j \sum_{k=1}^n \bar{Y}_k Z_k \right|^2 \\ &= \mathbb{E} \left(\sum_{i \neq j} (|Y_i|^2 |Z_j|^2 + Y_i \bar{Z}_i \bar{Y}_j Z_j + |EX_i^2|^2 Y_i Z_i \bar{Y}_j \bar{Z}_j) + \sum_{i=1}^n \mathbb{E}|X_i|^4 |Y_i|^2 |Z_i|^2 \right) \left| \sum_{k=1}^n \bar{Y}_k Z_k \right|^2 \\ &= \left(\sum_{i \neq k} (2(n-2)|Y_i|^2 |Y_k|^2 + (n-2)(|EZ_i^2|^2 + |EX_i^2|^2) Y_i^2 \bar{Y}_j^2) + \sum_{i=1}^n \mathbb{E}|X_i|^4 |Y_i|^4 \mathbb{E}|Z_i|^4 \right) \\ &\leq 2\kappa(n-2) \left(\sum_{i=1}^n |Y_i|^2 \right)^2 + \max_i \{ \mathbb{E}|X_i|^4 \mathbb{E}|Z_i|^4 - 2\kappa \} \sum_{i=1}^n |Y_i|^4,\end{aligned}$$

where $\kappa = 2$ for the real case and 1 for the complex case, X_i, Z_k are independent random variables with mean 0, variance 1 and finite 4th moment, and further $EX_i^2 = 0$ (and $EZ_i^2 = 0$) if they are complex, we will have

$$\begin{aligned}& \mathbb{E} \left| (\gamma_{j_1}^* \mathbf{B} \gamma_{j_2} \gamma_{j_1}^* \mathbf{B} \gamma_{k_1} \gamma_{k_2}^* \mathbf{B} \gamma_{j_2} \gamma_{j_2}^* \mathbf{B} \gamma_{k_1}) \right|^2 = \frac{1}{n} \mathbb{E} \left| (\gamma_{j_1}^* \mathbf{B} \gamma_{j_2} \gamma_{j_1}^* \mathbf{B} \gamma_{k_1} \gamma_{j_2}^* \mathbf{B} \gamma_{k_1}) \right|^2 \gamma_{j_2}^* \mathbf{B} \mathbf{T} \mathbf{B}^* \gamma_{j_2} \\ &\leq \frac{K}{n^4} \mathbb{E} (\gamma_{j_2}^* \mathbf{B} \mathbf{T} \mathbf{B}^* \gamma_{j_2})^3 + \frac{K}{n^5} \sum_{i=1}^n \mathbb{E} \left| \mathbf{e}_i' \mathbf{T}^{1/2} \mathbf{B} \gamma_{j_2} \right|^4 \gamma_{j_2}^* \mathbf{B} \mathbf{T} \mathbf{B}^* \gamma_{j_2} \\ &\leq \frac{K}{n^4} \left[\left(\frac{1}{n} (\mathbf{B} \mathbf{T} \mathbf{B}^* \mathbf{T}) \right)^3 + \frac{1}{n^3} \sum_{i=1}^n |\mathbf{e}_i' \mathbf{T}^{1/2} \mathbf{B} \mathbf{T} \mathbf{B}^* \mathbf{T}^{1/2} \mathbf{e}_i|^6 \mathbb{E}|X_{ij_2}|^6 \right] = O(n^{-4}),\end{aligned}$$

where \mathbf{e}_i is the standard i -th unit p -vector, i.e., its i -th entry is 1 and other $p - 1$ entries 0. In the last step of the above derivation, we have used facts that $\mathbb{E}|X_{ij_2}^6| \leq \eta_n^2 n \max \mathbb{E}|x_{ij}^4| = o(n)$ and $\mathbf{e}_i' \mathbf{T}^{1/2} \mathbf{B} \mathbf{T} \mathbf{B}^* \mathbf{T}^{1/2} \mathbf{e}_i \leq \|\mathbf{T}\|^2 / v^2$.

By similar approach, one can prove that the expectation of other products with the number of \mathbf{B} less than or equal to 6 are bounded by $O(n^{-3})$.

Case 2. There are 5 B's contained in the product. We shall use the first expansion of β_{j_1} and β_{j_2} and then use the bound bounded $|z|/v \leq K$ for β 's. Then we can show that such terms are also bounded by $O(n^{-3})$. Say, for the product of the first term of the first factor and the 6-th term of the second factor, its expectation is bounded by

$$\begin{aligned} & \left| \mathbb{E}(\beta_{j_1} \beta_{k_1(j_1)} \gamma_{j_1}^* \mathbf{B} \gamma_{k_1}) (\beta_{j_2} \beta_{k_2(j_2)} \beta_{k_1(j_2 k_2)} \beta_{j_1(j_2 k_1 k_2)}^2 \gamma_{j_2}^* \mathbf{B} \gamma_{j_1} \gamma_{j_1}^* \mathbf{B} \gamma_{k_1} \gamma_{k_1}^* \mathbf{B} \gamma_{j_1} \gamma_{j_1}^* \mathbf{B} \gamma_{k_2})^* \right| \\ &= \left| \mathbb{E} \left(\beta_{j_1} \beta_{k_1(j_1)} \beta_{j_2} \beta_{k_2(j_2)} \beta_{k_1(j_2 k_2)} \beta_{j_1(j_2 k_1 k_2)}^2 - b_{j_1} b_{k_1(j_1)} b_{j_2} 1_{k_2(j_2)} b_{k_1(j_2 k_2)} b_{j_1(j_2 k_1 k_2)}^2 \right) \times \right. \\ & \quad \left. \gamma_{j_1}^* \mathbf{B} \gamma_{k_1} \gamma_{j_2}^* \mathbf{B} \gamma_{j_1} \gamma_{j_1}^* \mathbf{B} \gamma_{k_1} \gamma_{k_1}^* \mathbf{B} \gamma_{j_1} \gamma_{j_1}^* \mathbf{B} \gamma_{k_2} \right)^* \Big| \\ &\leq K \left(\mathbb{E} \left| \left(\beta_{j_1} \beta_{k_1(j_1)} \beta_{j_2} \beta_{k_2(j_2)} \beta_{k_1(j_2 k_2)} \beta_{j_1(j_2 k_1 k_2)}^2 - b_{j_1} b_{k_1(j_1)} b_{j_2} 1_{k_2(j_2)} b_{k_1(j_2 k_2)} b_{j_1(j_2 k_1 k_2)}^2 \right) \right. \right. \\ & \quad \left. \left. (\gamma_{j_1}^* \mathbf{B} \gamma_{k_1} \gamma_{j_2}^* \mathbf{B} \gamma_{j_1}) \right|^2 \mathbb{E} \left| (\gamma_{j_1}^* \mathbf{B} \gamma_{k_1}) \right|^4 \left| \gamma_{j_1}^* \mathbf{B} \gamma_{k_2} \right|^2 \right)^{1/2} \leq O(n^{-3}). \end{aligned}$$

Here, we have used the fact that each term in the expansion of

$$\left(\beta_{j_1} \beta_{k_1(j_1)} \beta_{j_2} \beta_{k_2(j_2)} \beta_{k_1(j_2 k_2)} \beta_{j_1(j_2 k_1 k_2)}^2 - b_{j_1} b_{k_1(j_1)} b_{j_2} 1_{k_2(j_2)} b_{k_1(j_2 k_2)} b_{j_1(j_2 k_1 k_2)}^2 \right)$$

contains at least one ϵ function which the centralized quadratic form of γ . The use the same approach employed in Case1, one can show that the bound is $O(n^{-3})$.

Case 3. There are less than 5 B's contained in the product. If the number is 4, we need to further expand the matrix \mathbf{A}_{j_1} in ϵ_{j_1} as $\mathbf{A}_{j_1 j_2}^{-1} - \mathbf{A}_{j_1 j_2}^{-1} \gamma_{j_2} \gamma_{j_2}^* \mathbf{A}_{j_1 j_2}^{-1} \beta_{j_2(j_1)}$, expand $\mathbf{A}_{j_2}^{-1} = \mathbf{A}_{j_1 j_2}^{-1} - \mathbf{A}_{j_1 j_2}^{-1} \gamma_{j_1} \gamma_{j_1}^* \mathbf{A}_{j_1 j_2}^{-1} \beta_{j_1(j_2)}$ in ϵ_{j_2} , and then use the approach employed in Case 2 to obtain the desired bound.

If the number is less than 4, we need to further expand the inverses of \mathbf{A} -matrices. The details are omitted. Finally, we obtain that

$$\sum_{(4)} = O\left(\frac{1}{n}\right).$$

Similarly, we have

$$\sum_{(3)} = O\left(\frac{1}{n}\right).$$

Because $\text{tr}(\mathbf{A}^{-2} \Delta) = \frac{d}{dz} \text{tr}(\mathbf{A}^{-1} \Delta)$, then we have

$$\mathbb{E}|\text{tr}(\mathbf{A}^{-2} \Delta)|^2 = \mathbb{E} \left| \frac{1}{n} \sum_{j \neq k} \gamma_j^* \mathbf{A}^{-2} \gamma_k \right|^2 = O\left(\frac{1}{n}\right).$$

The lemma is proved.

Lemma 4.4 *After truncation and normalization, we have*

$$\text{tr}(\mathbf{A}^{-2}\Delta\mathbf{A}^{-1}\Delta) = (\underline{m}_y(z) + z\underline{m}'_y(z))(1 + z\underline{m}_y(z))$$

in L_2 uniformly for $z \in \mathbb{C}^+$.

Proof. Set $\text{tr}\mathbf{A}^{-1}(z_1)\Delta\mathbf{A}^{-1}(z_2)\Delta = \frac{1}{n^2} \sum_{j \neq k, i \neq t} \gamma_i^* \mathbf{A}^{-1}(z_1) \gamma_k \gamma_j^* \mathbf{A}^{-1}(z_2) \gamma_t = Q_1 + Q_2$ where

$$Q_1 = \frac{1}{n^2} \sum_{j \neq k} \gamma_j^* \mathbf{A}^{-1}(z_1) \gamma_j \gamma_k^* \mathbf{A}^{-1}(z_2) \gamma_k \quad \text{and} \quad Q_2 = \frac{1}{n^2} \sum_{\substack{j \neq k, i \neq t \\ i \neq k, \text{ or } j \neq t}} \gamma_i^* \mathbf{A}^{-1}(z_1) \gamma_k \gamma_j^* \mathbf{A}^{-1}(z_2) \gamma_t.$$

By Lemma 4.2 and 4.3, we obtain $E|Q_1 - (1 + z\underline{m}_y(z_1))(1 + z\underline{m}_y(z_2))|^2 \leq Kn^{-1}$ and $E|Q_2|^2 = o(1)$. We thus have $E|\text{tr}\mathbf{A}^{-1}(z_1)\Delta\mathbf{A}^{-1}(z_2)\Delta - (1 + z\underline{m}_y(z_1))(1 + z\underline{m}_y(z_2))|^2 = o(1)$. Consequently, because $\text{tr}\mathbf{A}^{-2}(z_1)\Delta\mathbf{A}^{-1}(z_2)\Delta = \frac{\partial \text{tr}\mathbf{A}^{-1}(z_1)\Delta\mathbf{A}^{-1}(z_2)\Delta}{\partial z_1}$, then we have $E|\text{tr}\mathbf{A}^{-2}(z_1)\Delta\mathbf{A}^{-1}(z_2)\Delta - \frac{\partial}{\partial z_1}g(z_1)g(z_2)|^2 = o(1)$. That is,

$$\text{tr}\mathbf{A}^{-2}(z_1)\Delta\mathbf{A}^{-1}(z_2)\Delta = g(z_2)g'(z_1) \text{ in } L_2.$$

By setting $z_1 = z_2 = z$, we obtain $\text{tr}(\mathbf{A}^{-2}\Delta\mathbf{A}^{-1}\Delta) = g(z)g'(z)$ in L_2 .

Lemma 4.5 *After truncation and normalization, we have $\text{tr}(\mathbf{A}^{-1}\Delta)^3(\mathbf{A} - \Delta)^{-1} = g(z)\text{tr}((\mathbf{A}^{-1}\Delta)^2(\mathbf{A} - \Delta)^{-1}) + o_p(1)$ uniformly for $z \in \mathbb{C}^+$.*

Proof. We have

$$\begin{aligned} \text{tr}(\mathbf{A}^{-1}\Delta)^3(\mathbf{A} - \Delta)^{-1} &= E \frac{1}{n^3} \sum_{\substack{i \neq t, j \neq g \\ h \neq s}} \gamma_i^* \mathbf{A}^{-1} \gamma_j \gamma_g^* \mathbf{A}^{-1} \gamma_h \gamma_s^* (\mathbf{A} - \Delta)^{-1} \mathbf{A}^{-1} \gamma_t \\ &= \frac{1}{n^3} \sum_{\substack{i \neq t, j \neq g \\ i=j, h \neq s}} \gamma_i^* \mathbf{A}^{-1} \gamma_j \gamma_g^* \mathbf{A}^{-1} \gamma_h \gamma_s^* (\mathbf{A} - \Delta)^{-1} \mathbf{A}^{-1} \gamma_t + \frac{1}{n^3} \sum_{\substack{i \neq t, j \neq g \\ i \neq j, h \neq s}} \gamma_i^* \mathbf{A}^{-1} \gamma_j \gamma_g^* \mathbf{A}^{-1} \gamma_h \gamma_s^* (\mathbf{A} - \Delta)^{-1} \mathbf{A}^{-1} \gamma_t \\ &= g(z) \frac{1}{n^2} \sum_{h \neq s} \gamma_g^* \mathbf{A}^{-1} \gamma_h \gamma_s^* (\mathbf{A} - \Delta)^{-1} \mathbf{A}^{-1} \gamma_t + o_p(1) \\ &= g(z) \text{tr}((\mathbf{A}^{-1}\Delta)^2(\mathbf{A} - \Delta)^{-1}) + g(z) \frac{1}{n^2} \sum_{\substack{g=t \\ h \neq s}} \gamma_g^* \mathbf{A}^{-1} \gamma_h \gamma_s^* (\mathbf{A} - \Delta)^{-1} \mathbf{A}^{-1} \gamma_t + o_p(1) \\ &= g(z) \text{tr}((\mathbf{A}^{-1}\Delta)^2(\mathbf{A} - \Delta)^{-1}) + o_p(1). \end{aligned}$$

Then by Lemma 4.4, we have

$$\begin{aligned} \text{tr}(\mathbf{A}^{-1}\Delta)^2(\mathbf{A} - \Delta)^{-1} &= \text{tr}(\mathbf{A}^{-1}\Delta)^2(\mathbf{A})^{-1} + \text{tr}(\mathbf{A}^{-1}\Delta)^3(\mathbf{A} - \Delta)^{-1} \\ &= \text{tr}(\mathbf{A}^{-1}\Delta)^2(\mathbf{A})^{-1} + g(z) \text{tr}(\mathbf{A}^{-1}\Delta)^2(\mathbf{A} - \Delta)^{-1} + o_p(1) \\ &= \frac{(1 + z\underline{m}_y(z))(\underline{m}_y(z) + z\underline{m}'_y(z))}{1 - g(z)} + o_p(1). \end{aligned}$$

Hence, we obtain the following lemma.

Lemma 4.6 *After truncation and normalization, we have*

$$\text{tr} \mathbf{A}^{-2}(z) \Delta + \text{tr} \mathbf{A}^{-1}(z) (\Delta \mathbf{A}^{-1}(z))^2 + \text{tr} (\mathbf{A}(z) - \Delta)^{-1} (\Delta \mathbf{A}^{-1}(z))^3 = \frac{(\underline{m}_y(z) + z \underline{m}'_y(z))(1 + z \underline{m}_y(z))}{-z \underline{m}_y(z)} + o_p(1)$$

uniformly for $z \in \mathbb{C}^+$.

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